High-Order Curvilinear Element Methods in Lake Modelling

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Introduction

Simulating the motion of lakes is a complicated problem due to the wide variety of laminar and turbulent flow features of length scales ranging from 1 km to 1 mm. The difficulty is further compounded by the irregular geometry of typical lake coastlines.

In order to deal with general geometries, we investigate the use of the high-order nodal Discontinuous Galerkin Finite Element Method (DG-FEM) to solve a weakly non-hydrostatic layered model [1] that includes dispersive terms to prevent unphysical shocks from forming and to model high-frequency wave phenomena. For a single-layer fluid, the equations are:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \nabla\cdot\begin{pmatrix} hu & hv \\ hu^2+\frac{1}{2}gh^2 & huv \\ huv & hv^2+\frac{1}{2}gh^2 \end{pmatrix} = gh\begin{pmatrix} 0 \\ \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} + f\begin{pmatrix} 0 \\ vh \\ -uh \end{pmatrix} + \frac{H^2}{6}\begin{pmatrix} 0 \\ \frac{\partial}{\partial x}\nabla\cdot(h\mathbf{u})_t \\ \frac{\partial}{\partial y}\nabla\cdot(h\mathbf{u})_t \end{pmatrix}$$

Here, $h(x,y,t) = H(x,y) + \eta(x,y,t)$ represents the total layer thickness where $H(x,y)$ is the mean depth. The parameters g and f are the acceleration due to gravity and the Coriolis parameter. The terms on the right-hand side (from left to right) represent variations in depth, Coriolis accelerations, and weakly non-hydrostatic corrections to the hydrostatic pressure gradient.

In the nodal DG-FEM method, the local solution to the weak-integral form of the equations are considered on a particular element with (Lagrange) interpolatory test functions. Inter-element coupling is handled using suitable numerical flux functions in the surface integral contributions. For computational purposes, the integral formulations are reduced to local (mass & stiffness matrix) operators. See [2] for details.

Motivation

Even a very smooth boundary may appear to have sharp corners when approximated in a piece-wise linear manner, as is the case with most traditional finite element approximations. This fact represents a source of two key difficulties:

1. It is well known that the convergence rate of high-order methods is a function of the smoothness of boundaries. As a result, it is often argued that in the presence of non-smooth boundaries, a first-order approximation is the best one can do.

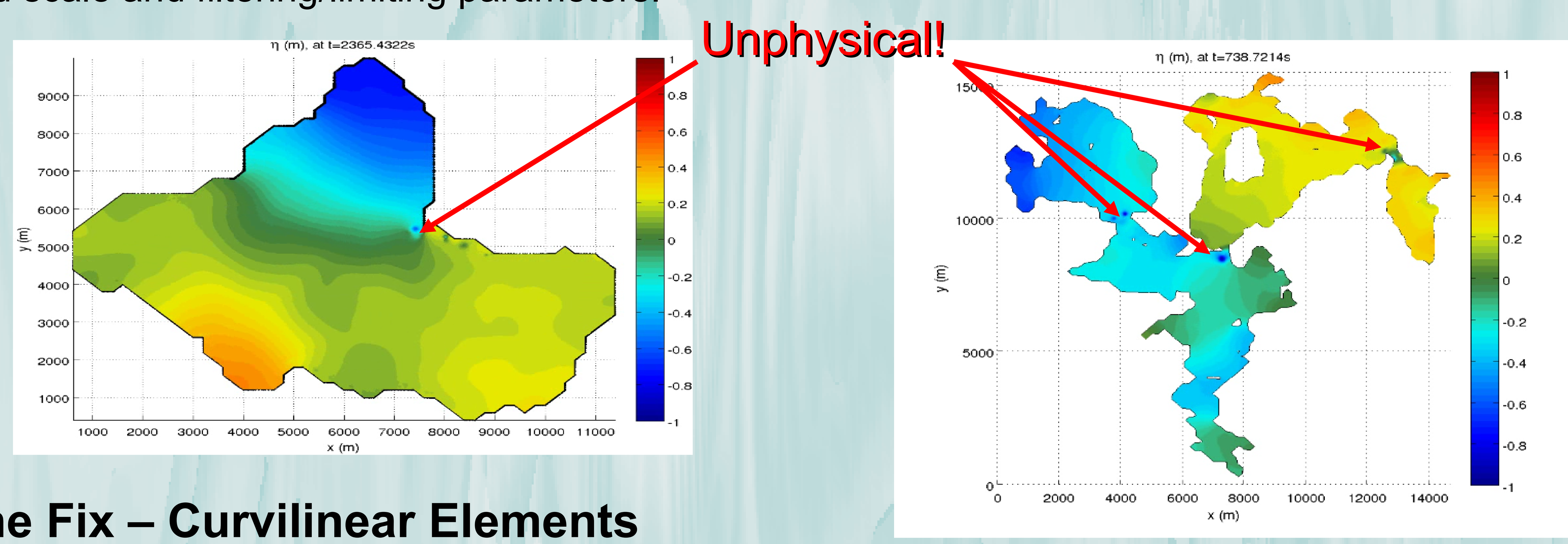
2. Inviscid flow around a wall corner contains a singularity at the corner whenever the wall angle is >180°, as can be demonstrated using potential flow theory [3].

In light of these issues, it is clear that a high-order method begs for a smooth and highly accurate representation of coastlines.

Spurious Eddies in Inviscid Simulations

It has been found that in numerical simulations, the aforementioned singularity at corners may be tamed using limiting or local spectral filtering. However, these techniques have the rather undesired effect of producing spurious eddies by diffusing the singularity outward from the boundary.

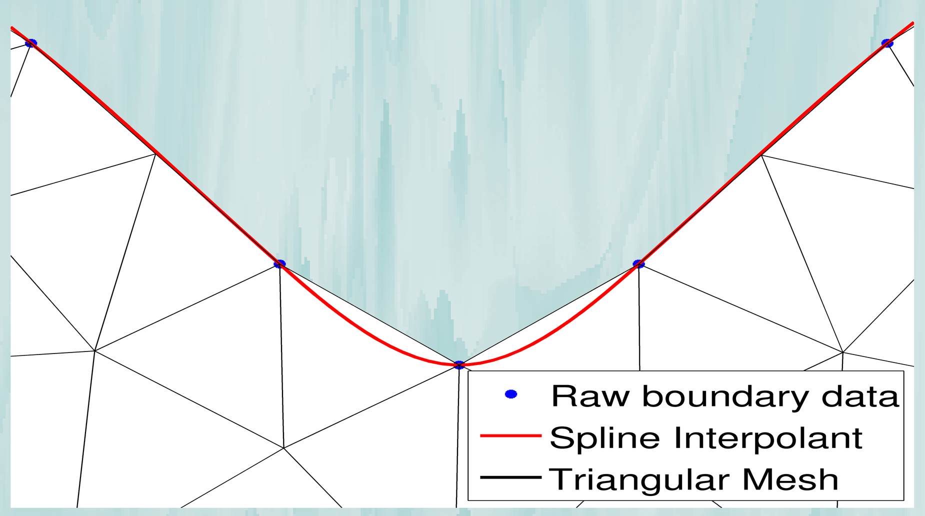
Since the equations do not contain viscous terms, viscous boundary layers are not modelled. Thus, these eddies are unphysical, and it can be shown that their shape and size are dependent on the grid scale and filtering/limiting parameters.



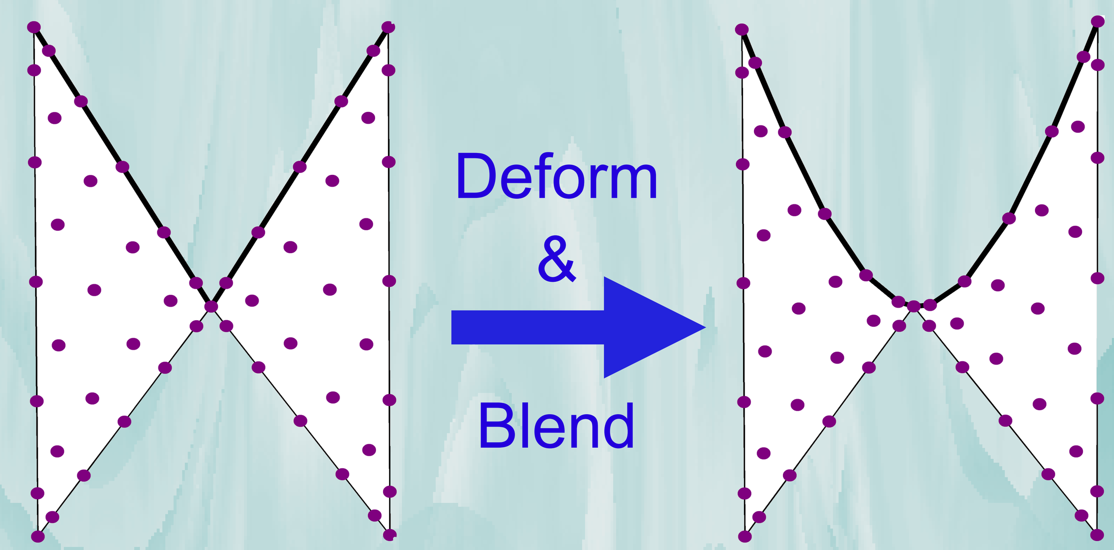
The Fix – Curvilinear Elements

We have extended the procedure presented in [2] for circular geometries to deal with arbitrary curvilinear geometries as follows:

1. Given unstructured boundary data, construct an arc length parameterized cubic spline interpolant that may be used to smoothly represent/reconstruct the coastlines.



2. Generate a straight-sided triangular mesh using the original (or sub-sampled) unstructured boundary data and your favorite mesh generator.



3. Flag all boundary element edges that need to be curved. In each flagged element, evaluate the parametric spline to redistribute the edge nodes by arc length along the curvilinear boundary.

4. Calculate the deformation in moving the boundary nodes from the straight-sided edge to the curvilinear edge. The deformation can then be blended to interior nodes using Gordon-Hall blending [2]. The Jacobian of the mapping to the standard triangle can then be computed numerically.

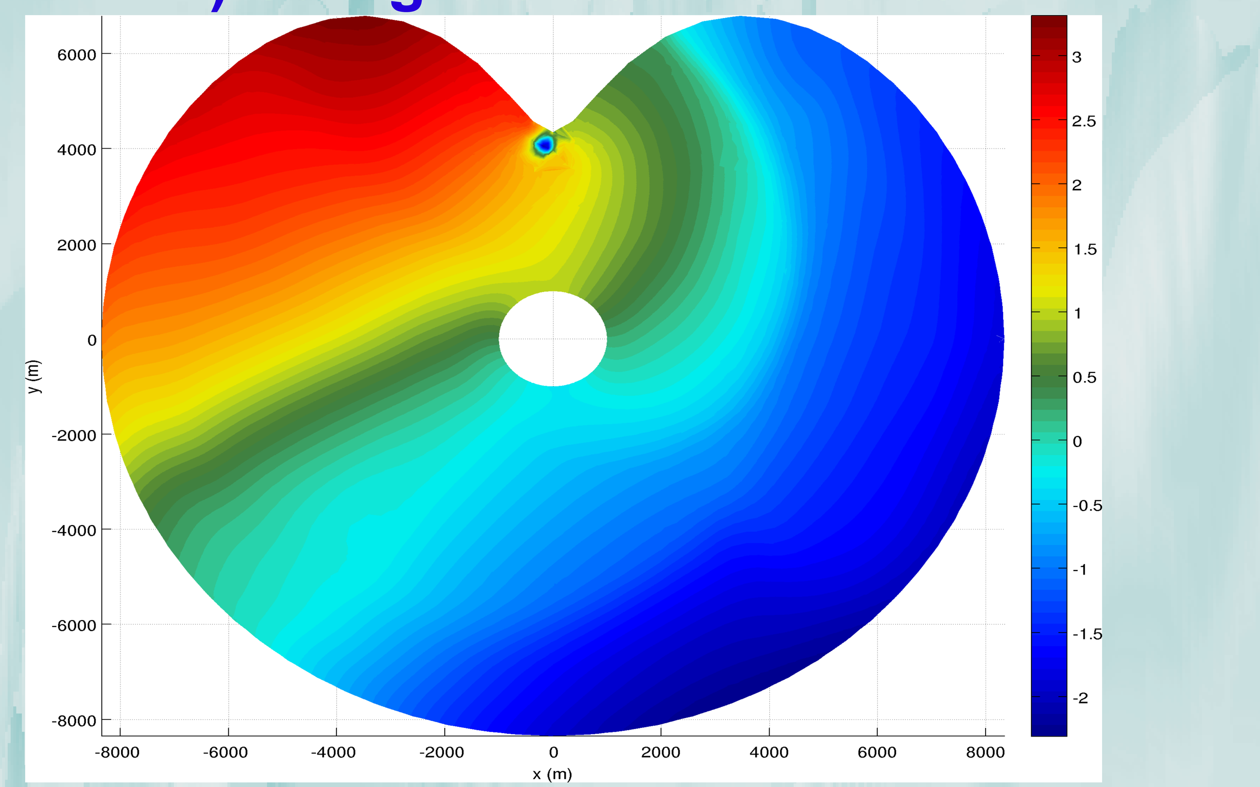
Hence, the local mass and stiffness matrices can be recovered on all curvilinear elements. These local operators can then be stored (and factorized, if necessary) for re-use at each time-step.

In order to reduce the effects of aliasing errors induced by the non-constant Jacobians, quadrature & cubature integration rules of higher order than the interpolating polynomials are employed, as in [2].

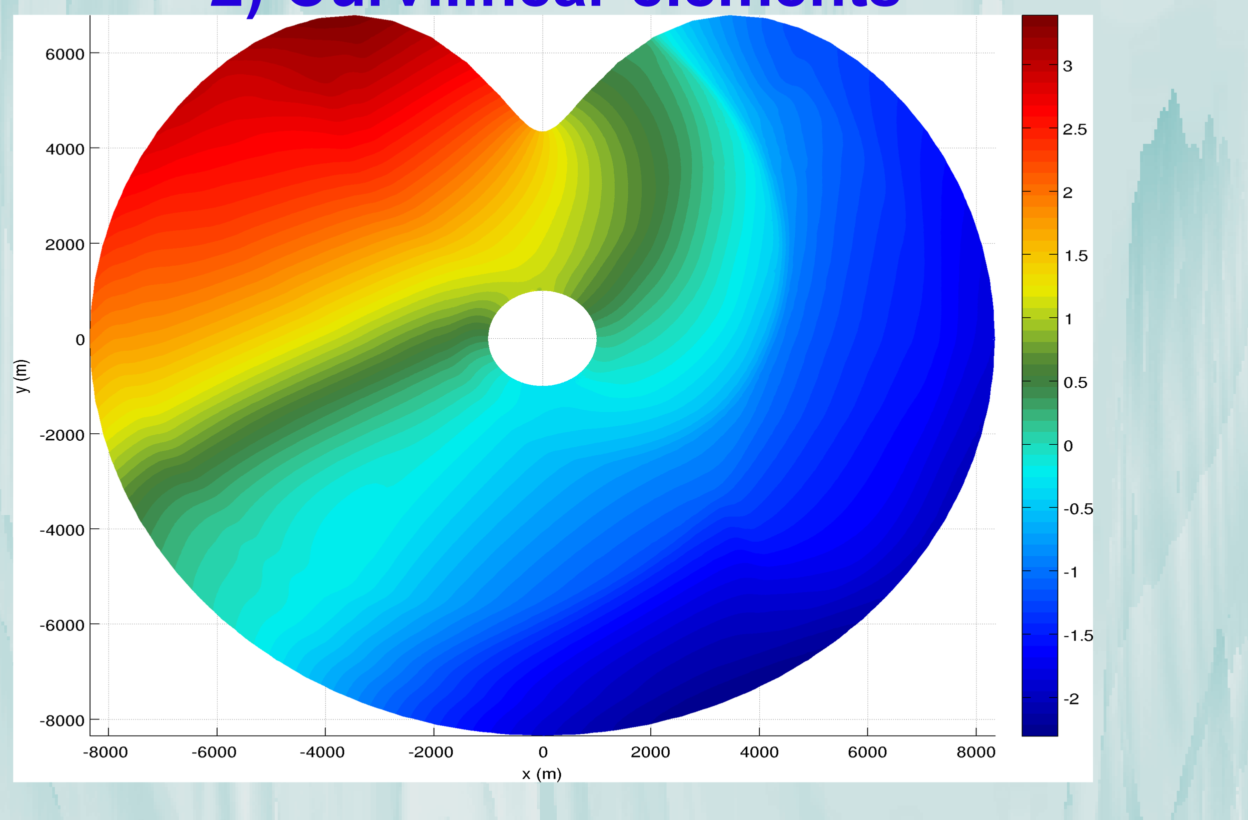
Results and Discussion

Preliminary results on simplified geometries reveal that the curvilinear element methodology suppresses spurious eddies at the smoothed corners, as shown below using an 8th order simulation of an evolving density interface tilt in a rotating basin.

1) Straight-sided elements



2) Curvilinear elements



Another benefit of using the over-integration techniques comes from the observation that less filtering is required to stabilize the scheme due to a reduction of aliasing errors. Hence, solutions are obtained with limited amounts of numerical dissipation.

It was found that the switch to the curvilinear element methodology resulted in an increase in computational time by a factor of ~1.4 over the straight-sided element methodology. Thus, the overall outlook for applying the curvilinear element approach in practical applications is quite promising.

References

[1] A. de la Fuente, K. Shimizu, J. Imberger, and Y. Nino. The evolution of internal waves in a rotating, stratified, circular basin and the influence of weakly nonlinear and nonhydrostatic accelerations. *Limnol. Oceanogr.*, **53**(6):2738–2748, 2008.
[2] J. Hesthaven and T. Warburton. Nodal Discontinuous Galerkin Methods. Springer, 2008.
[3] P. Kundu and I. Cohen. Fluid Mechanics, 4th edition. Elsevier Academic Press, 2008.