Calculating free modes of oscillation in complex geometries

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I. Introduction

The linearized shallow water equations on the rotating f-plane can be written as

$$\frac{\partial \mathbf{M}}{\partial t} + f\mathbf{M}^{\perp} = -gH\nabla\eta \tag{1}$$

$$\frac{\partial \eta}{\partial t} + \nabla\cdot\mathbf{M} = 0 \tag{2}$$

where $H(\mathbf{x})$ is allowed to vary in space in terms of the Cartesian position vector $\mathbf{x} = (x, y)$, $\eta(\mathbf{x}, t)$ is the free surface elevation above the undisturbed state, $\mathbf{M} = H\mathbf{u}$ is the volume transport vector, $\mathbf{u} = (u(\mathbf{x}, t), v(\mathbf{x}, t))$ is the fluid velocity field, and superscript $\perp$ denotes rotation by 90° counter-clockwise, so that $\mathbf{M}^{\perp} = H(-v, u)$. The parameters g and f represent the acceleration due to gravity and the Coriolis parameter, respectively. The fluid depth is non-dimensionalized via $H(\mathbf{x}) = \overline{H}h(\mathbf{x})$, where $\overline{H}$ is the mean basin depth and $h(\mathbf{x})$ is dimensionless.

If $f = 0$, the equations can be combined to yield

$$\eta_{tt} = g\overline{H}\nabla\cdot h\nabla\eta \,. \tag{3}$$

Looking for solutions that are periodic in time, i.e.,

$$\eta = \hat{\eta}e^{i\sigma t} \,, \tag{4}$$

yields the familiar Laplace eigenvalue problem

$$\nabla\cdot h\nabla\hat{\eta} = \lambda\hat{\eta} \,, \quad \text{on} \quad \Omega \,, \tag{5}$$

where $\lambda = -\sigma^2/g\overline{H}$, subject to the boundary condition

$$\frac{\partial\hat{\eta}}{\partial n} = 0 \,, \quad \text{on} \quad \partial\Omega \,. \tag{6}$$

On the other hand, if $f \neq 0$, combining the equations results in

$$\nabla\cdot h\nabla\hat{\eta} + \left(\frac{\sigma^2 - f^2}{g\overline{H}}\right)\hat{\eta} + \frac{if}{\sigma}\nabla h\cdot\nabla^{\perp}\eta = 0 \,, \quad \text{on} \quad \Omega \,, \tag{7}$$

subject to the boundary condition

$$\frac{\partial\eta}{\partial n} - \frac{if}{\sigma}\frac{\partial\eta}{\partial s} = 0 \,, \quad \text{on} \quad \partial\Omega \,. \tag{8}$$

Upon inspecting the eigenvalue problem, we notice that unlike the Laplace eigenvalue problem, the operator is not self-adjoint. Further difficulty results from the fact that the no normal flow boundary condition in terms of η contains real and imaginary components and is coupled to the eigenvalue, σ.

II. Methods

To circumvent the difficult nature of the eigenvalue problem (7)–(8), we invoke the Helmholtz decomposition

$$\mathbf{M} = -h\nabla\phi + \nabla^{\perp}\psi \,, \tag{9}$$

where ϕ and ψ satisfy the simpler boundary conditions

$$\frac{\partial\phi}{\partial n} = \psi = 0 \quad \text{on} \quad \partial\Omega \,. \tag{10}$$

The next step is to form two sets of basis functions based on two eigenvalue problems. We define the set $\{\phi_\alpha, \lambda_\alpha\}_{\alpha=1}^{N_\phi}$ given by the eigenproblem

$$\nabla\cdot(h\nabla\phi_\alpha) + \lambda_\alpha\phi_\alpha = 0 \,, \quad \text{on} \quad \Omega \,, \tag{11}$$

$$\nabla\phi_\alpha\cdot\hat{\mathbf{n}} = 0 \,, \quad \text{on} \quad \partial\Omega \,, \tag{12}$$

and the set $\{\psi_\alpha, \mu_\alpha\}_{\alpha=1}^{N_\psi}$ given by

$$\nabla\cdot\left(\frac{1}{h}\nabla\psi_\alpha\right) + \mu_\alpha\psi_\alpha = 0 \,, \quad \text{on} \quad \Omega \,, \tag{13}$$

$$\psi_\alpha = 0 \,, \quad \text{on} \quad \partial\Omega \,. \tag{14}$$

Here, N_ϕ and N_ψ are taken as finite integers to truncate the bases for computational reasons. It can be shown using dimensional analysis that a suitable normalization for the basis functions is

$$\iint_\Omega h\nabla\phi_k\cdot\nabla\phi_l dA = \lambda_l\iint_\Omega \phi_k\phi_l dA = Ac^2\overline{H}^2\delta_{k,l} \,, \tag{15}$$

$$\iint_\Omega \frac{1}{h}\nabla^{\perp}\psi_k\cdot\nabla^{\perp}\psi_l dA = \mu_l\iint_\Omega \psi_k\psi_l dA = Ac^2\overline{H}^2\delta_{k,l} \,. \tag{16}$$

Next, ϕ and ψ are expanded in terms of the basis functions via

$$\phi = \sum_\alpha P_\alpha(t)\phi_\alpha \,, \tag{17}$$

$$\psi = \sum_\alpha Q_\alpha(t)\psi_\alpha \,. \tag{18}$$

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Now, since

$$\eta_t = -\nabla \cdot \mathbf{M} = \nabla \cdot (h\nabla\phi) = -\sum_\alpha P_\alpha \lambda_\alpha \phi_\alpha , \quad (19)$$

it is clear that η should be expanded in terms of the ϕ_α's. Rao and Schwab (1976) set

$$\eta = \sum_\alpha R_\alpha(t)\eta_\alpha = \sum_\alpha R_\alpha(t)\frac{\lambda_\alpha^{\frac{1}{2}}}{c}\phi_\alpha , \quad (20)$$

such that the R_α's are dimensionless, just as the P_α's and Q_α's are, by construction.

A system of ordinary differential equations for the expansion coefficients can be found by substituting the expansions (17)–(20) into the momentum and continuity equations, taking appropriate inner products, and exploiting orthogonality resulting in a system of the form

$$\frac{d}{dt}\mathbf{V} + E\mathbf{V} = 0 , \quad (21)$$

where

$$\mathbf{V} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} , \quad \text{and} \quad E = \begin{pmatrix} -A & -B & -<\nu> \\ -C & -D & 0 \\ <\nu> & 0 & 0 \end{pmatrix} , \quad (22)$$

and

$$<\nu> = \begin{pmatrix} \nu_1 & 0 & \dots & 0 \\ 0 & \nu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_{N_\phi} \end{pmatrix} = \text{diag}\,(\nu_i) \ . \quad (23)$$

Here,

$$A_{\beta\alpha} = -\frac{1}{Ac^2\overline{H}^2}\iint_\Omega fh\nabla\phi_\beta \cdot \nabla^\perp\phi_\alpha \, dA , \quad (24)$$

$$B_{\beta\alpha} = -\frac{1}{Ac^2\overline{H}^2}\iint_\Omega f\nabla\psi_\alpha \cdot \nabla\phi_\beta \, dA , \quad (25)$$

$$\nu_\beta = c\lambda_\beta^{\frac{1}{2}} . \quad (26)$$

$$\begin{aligned} C_{\beta\alpha} &= \frac{1}{Ac^2\overline{H}^2}\iint_\Omega f\nabla^\perp\phi_\alpha \cdot \nabla^\perp\psi_\beta \, dA , \\ &= \frac{1}{Ac^2\overline{H}^2}\iint_\Omega f\nabla\phi_\alpha \cdot \nabla\psi_\beta \, dA , \\ &= -B_{\alpha\beta} . \end{aligned} \quad (27)$$

and

$$D_{\beta\alpha} = \frac{1}{Ac^2\overline{H}^2}\iint_\Omega f\frac{1}{h}\nabla\psi_\alpha \cdot \nabla^\perp\psi_\beta . \quad (28)$$

Assuming

$$\mathbf{V}(t) = e^{i\sigma t}\hat{\mathbf{V}} , \quad (29)$$

we recover a matrix eigenvalue problem of the form

$$iE\hat{\mathbf{V}} = \sigma\hat{\mathbf{V}} . \quad (30)$$

Inspecting the structure of E and recalling that the matrices A and D are symmetric, $B = -C^T$, it follows that E is antisymmetric. Therefore, iE is Hermitian and the eigenvalues σ are real. Furthermore, since iE is purely imaginary and σ is purely real, it follows that all eigenvectors must have real and imaginary components in order to satisfy the eigenproblem (30). Physically, this property corresponds to the various basis functions being out of phase with one another.

III. Benchmark Results: Kelvin waves in a circular basin

As a means of validating our numerical method, we have reproduced the analytical calculations performed by Csanady (1967) for a two-layer flat-bottomed circular basin of radius $r_0 = 67.5$ km representing a model great lake. The acceleration due to gravity and Coriolis parameter were taken to be $g = 9.81$ ms^{-2}, $f = 10^{-4}$ s^{-1}, respectively. The upper-layer and lower-layer thicknesses were taken to be $H_1 = 15$ m and $H_2 = 60$ m, and the density jump between the upper and lower layers was taken to be $\Delta\rho = 1.74$ kg m^{-3} with a reference density of $\rho_0 = 1000$ kg m^{-3}. The calculations follow the normal modes decomposition in the vertical direction Csanady (1967), so that the barotropic (surface) horizontal free modes of oscillation are computed using the long wave speed for surfaces wave $c_{bt} = \sqrt{gH}$ where $H = H_1 + H_2$ is the total depth, and the baroclinic (internal, vertical mode 1) horizontal modes are computed using the long internal wave speed $c_{bc} = \sqrt{gH_e}$ where $H_e = (\Delta\rho/\rho_0)\,H_1H_2/H$ is the equivalent depth. The importance of the density stratification relative to the Earth's rotation is captured by the non-dimensional quantity

$$S = \frac{c}{fr_0} . \quad (31)$$

	Current Method	Analytical	
s	σ/f	σ/f	Relative Error
1	0.069514	0.069418	0.0013844
2	0.13904	0.13883	0.0015106
3	0.20853	0.20824	0.0013735
4	0.27807	0.27764	0.0015312
5	0.34752	0.34703	0.0014061
6	0.41709	0.41641	0.0016186

Table 1: *Analytical and numerical values for the non-dimensionalized frequencies (σ/f) of Kelvin modes with azimuthal mode number s in a stratified rotating basin with Burger number $S = 0.067$.*

Sample animations and discussion follows.