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High-Order Methods for Weakly Non-Hydrostatic Layered Models

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The Model

The wavelength constraint on the traditional Shallow Water Model (SWM) requires wavelengths to be much larger than the depth

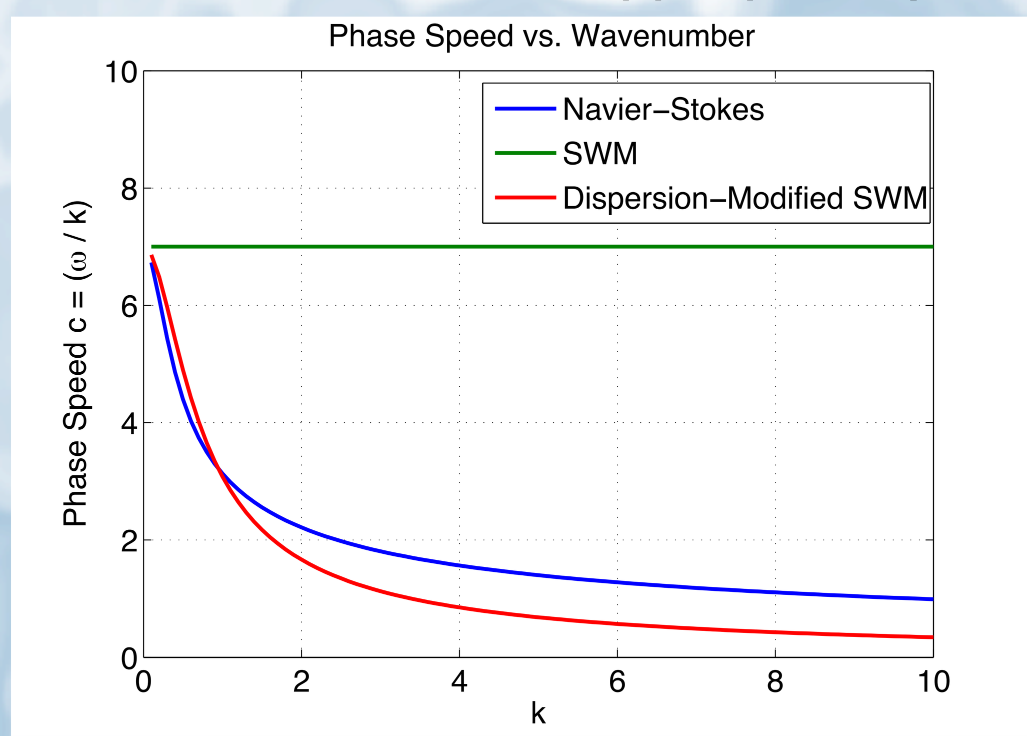
$$(H/\lambda) \ll 1\,.$$

This constraint may be relaxed to allow for weakly non-hydrostatic corrections to the hydrostatic pressure, yielding the following dispersion-modified SWM for a single fluid layer of constant density

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -g\nabla\eta + f(\mathbf{u}\times\hat{\mathbf{k}}) + (H^2/6)\nabla(\nabla\cdot\mathbf{u}_t)\,,$$

$$\eta_t + \nabla\cdot((H+\eta)\mathbf{u}) = 0\,,$$

that can be derived by retaining only terms of order $(H/L)^2\,,(a/H)$in the so-called Boussinesq equations [Brandt1997]. With the added dispersion terms, dispersive short-waves are now appropriately modelled.



In this study, we use Fourier and Chebyshev pseudospectral (global, high-order) spatial discretization methods to solve the above equations. Due to the presence of a time-derivative in the dispersion terms, explicit time-stepping formulas for the velocity fail to be stable, and the time-stepping problem must be written in terms of a 2x2 block linear system.

In the case of a flat bottom (H=const.), FFT-based solvers provide an efficient means of solving the linear system described above. However, for non-constant depth, similar solvers do not exist, and solution by iteration is required. The dimension of the required linear system may be reduced by a factor of 2 by forming an auxiliary equation for the non-hydrostatic pressure variable $(z = \nabla\cdot\mathbf{u}_t)$ [Karniadakis2005]

$$\nabla\cdot((H^2/6)\nabla z) - z = \nabla\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u} + g\nabla\eta + f(\hat{\mathbf{k}}\times\mathbf{u})]\,,$$

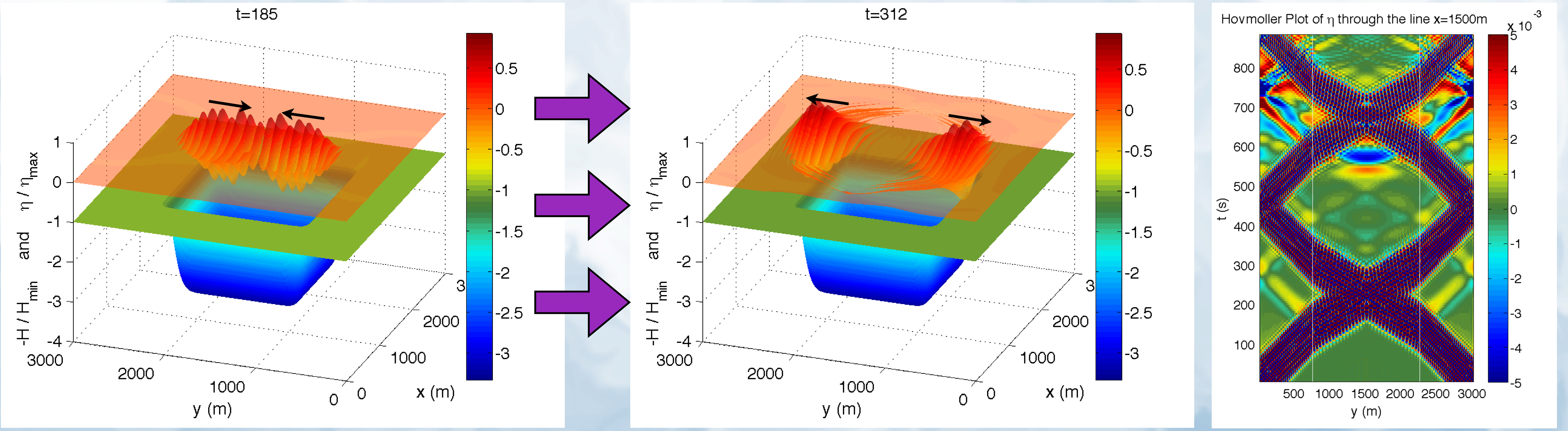
which must be solved for z at each time-level.

In addition to giving the highest order of spatial accuracy possible, the numerical method employed is also very non-dissipative. The only dissipation takes the form of a tuneable low-pass wavenumber filter that removes energy from the unphysical small scales.

Results

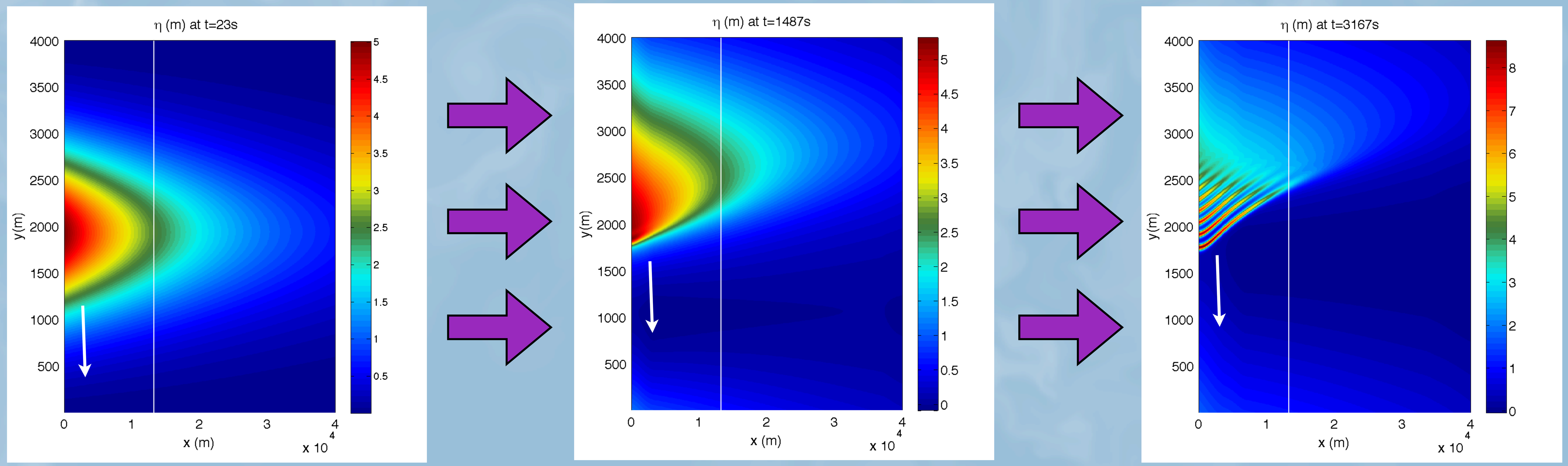
Short Wave Interactions with Topography

An initial packet of short waves centered at the periodic boundary was released with no initial velocity. As the two wave-trains pass into the deep region, their phase speeds increase and their amplitudes decrease. As they exit the deep region and enter the shallow region, the opposite effect occurs thereby enhancing the dispersion and nonlinearity which reorders the wave-trains. Topographical wave refraction is another apparent feature of the simulation. Model resolution is 1024x1024.



Dispersive Break-Down of a Nonlinear Kelvin Wave

The nonlinear Kelvin wave obtained from the traditional SWM steepens and forms a shock. Our recent simulations with the dispersion-modified SWM suggest that dispersion may act to redistribute the energy amongst a collection of high-energy solitary-like waves. Similar results were reported in [delaFuente2008]. The model was initialized with a large-amplitude surface perturbation against the western wall propagating southward with the linear long-wave speed. Relevant physical parameters: $f = 1.5\text{e-}4\ \mathrm{s}^{-1},\ g' = 0.2\ \mathrm{m\,s}^{-2},\ H = 20\ \mathrm{m}\,.$ The distance from the western wall to the white line represents the Rossby deformation radius. The channel is periodic in y (Fourier basis), and a Chebyshev basis is used in x. Model resolution is 512x1024.



Cited Literature

Brandt, P., A. Rubino, W. Alpers, J. Backhaus: "Internal waves in the Strait of Messina studied by a numerical model and synthetic aperture radar images from ERS 1/2 satellites," *J. Phys. Oceanogr.*, **27**, 648-663, 1997.
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de la Fuente, A., K. Shimizu, J. Imberger, Y. Nino: "The evolution of internal waves in a rotating, stratified, circular basin and the influence of weakly nonlinear and nonhydrostatic accelerations," *Limnol. Oceanogr.*, **53**(6), 2738-2748, 2008.